THE TEMPERATURE FIELD OF A TWO-BODY SYSTEM WITH A HEAT SOURCE

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We have derived formulas for the determination of the temperature field in two-layer bodies exhibiting various thermophysical properties and with a heat source whose intensity varies arbitrarily.

Let us consider a system consisting of a semiinfinite body and a plate with various thermophysical properties. The heat source is distributed uniformly over the plate surface. The heat flow $\vartheta(t)$ varies arbitrarily as specified.

The temperature field $T(z, t)$ is described by the equations

$$\frac{\partial^2 T_1}{\partial z^2} = \frac{1}{a_1}\frac{\partial T_1}{\partial t} \quad (0 \leqslant z \leqslant h), \tag{1}$$

$$\frac{\partial^2 T_2}{\partial z^2} = \frac{1}{a_2}\frac{\partial T_2}{\partial t} \quad (h \leqslant z < \infty) \tag{2}$$

with the initial and boundary conditions

$$T_1(z, 0) = T_2(z, 0), \tag{3}$$

$$T_1(h, t) = T_2(h, t), \tag{4}$$

$$\lambda_1 \frac{\partial T_1(h, t)}{\partial z} = \lambda_2 \frac{\partial T_2(h, t)}{\partial z}, \tag{5}$$

$$T_2(\infty, t) = 0. \tag{6}$$

The heat transferred from the source of the system at any instant of time is expended on the heating of the system, i.e.,

$$\int_0^t \vartheta(t)\,dt = c_1\gamma_1 \int_0^h T_1(z, t)\,dz + c_2\gamma_2 \int_h^\infty T_2(z, h)\,dz. \tag{7}$$

The thermophysical coefficients of the system are assumed to be independent of temperature. The exact solutions satisfying (3)-(7) will then be the following.

In the plate $0 \le h \le \infty$

$$T_1(z, t) = \frac{1}{\sqrt{\pi}\,\xi_1} \int_0^t \frac{\vartheta(\tau)}{\sqrt{t-\tau}} \left\{ \sum_{k=0}^{\infty} (-\varkappa)^k \exp\left[-\frac{(2hk+z)^2}{a_1(t-\tau)}\right] + \sum_{k=0}^{\infty} (-\varkappa)^{k+1} \exp\left[-\frac{[2h(k+1)-z]^2}{a_1(t-\tau)}\right]\right\} d\tau, \tag{8}$$

in the body $h \le z \le \infty$

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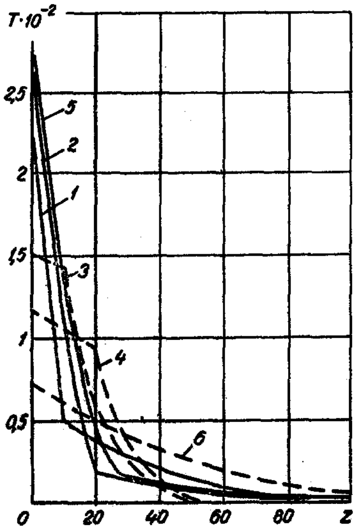

Fig. 1

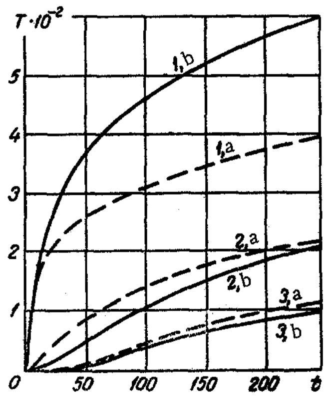

Fig. 2

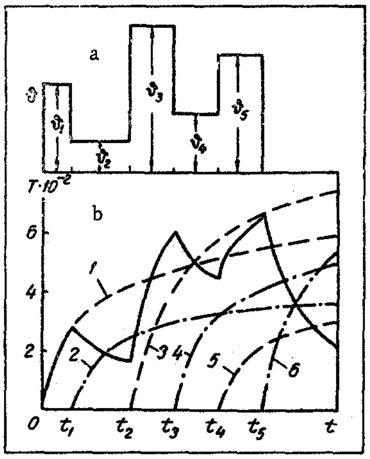

Fig. 3

Fig. 1. Temperature field of the following systems: 1) graphite plate h = 10 μm and copper body; 2) the same, h = 20 μm; 3) copper plate h = 10 μm and graphite body; 4) the same, h = 20 μm; 5) graphite body without plate; 6) copper body without plate; in the calculations we assume that: ϑ = 1 cal/cm^2·sec (for curves 5 and 6 we have ϑ = 0.5), t = 25 μsec, respectively, for the copper and the graphite we have c = 0.094 and 0.4 cal/g·deg, γ = 8.93 and 2 g/cm^3, λ = 0.72 and 0.05 cal/cm·sec·deg, a = 0.86 and 0.0625 cm^2/sec.

Fig. 2. Temperature as a function of time in a system made up of a graphite plate and a copper body: 1) at the plate surface; 2) at the boundary between the plate and the body; 3) in the body, at a depth of 100 μm from the plate surface; a) h = 10; b) 20 μm, ϑ = 1 cal/cm^2·sec.

Fig. 3. Rake-shaped heat source (a) and temperature as a function of time at the surface of a graphite plate 20 μm thick, with a copper body for ϑ = 1 (b). The dashed lines show the temperature from the negative sources. The solid line represents the overall temperature. The numbers at the curves correspond to the source numbers.

$$T_2(z, t) = \frac{2}{\sqrt{\pi}\,(\xi_1+\xi_2)} \int_0^t \frac{\vartheta(\tau)}{\sqrt{t-\tau}} \sum_{k=0}^{\infty} (-\varkappa)^k \exp\left[-\frac{\left[\dfrac{h(2k+1)}{\sqrt{a_1}} + \dfrac{z-h}{\sqrt{a_2}}\right]^2}{t-\tau}\right] d\tau, \tag{9}$$

$$\xi = \sqrt{c\gamma\lambda}, \quad \varkappa = \frac{\xi_2-\xi_1}{\xi_2+\xi_1}. \tag{10}$$

These last expressions describe the temperature field for any variation in the heat flow.

If ϑ = const, the temperature field is described by simpler formulas. In this case, having integrated (8) and (9), we find

$$T_1(z, t) = \frac{2\vartheta\sqrt{t}}{\sqrt{\pi}\,\xi_1} \sum_{k=0}^{\infty} (-\varkappa)^k \left[I(m_1) - \varkappa I(m_2)\right], \tag{11}$$

$$T_2(z, t) = \frac{4\vartheta\sqrt{t}}{\sqrt{\pi}\,(\xi_1+\xi_2)} \sum_{k=0}^{\infty} (-\varkappa)^k I(m_3), \tag{12}$$

$$I(m) = \exp(-m^2) - \sqrt{\pi}\,m\left[1 - \Phi(m)\right], \tag{13}$$

$$\Phi(m) = \frac{2}{\sqrt{\pi}} \int_0^m \exp(-y^2)\,dy \text{ — in the Kramp function,}$$

$$m_1 = \frac{2hk + z}{2\sqrt{a_1 t}}, \tag{14}$$

$$m_2 = \frac{2h(k+1) - z}{2\sqrt{a_1 t}}, \tag{15}$$

$$m_3 = \frac{h(2k+1)}{2\sqrt{a_1 t}} + \frac{z - h}{2\sqrt{a_2 t}}. \tag{16}$$

The function I(m) is strongly dependent on m. Since it diminishes sharply with an increase in m, the sums in (11) and (12) converge rather rapidly. In most cases, 2-3 terms are sufficient.

Figure 1 shows the temperature distribution in a copper–graphite system (plate thickness h = 10 and 20 μm, duration of heat-source operation 25 μsec, $\vartheta = 1$). For comparison, we present the temperature distribution in a one-component system of the same materials.

The temperature at the surface of the plate and at the boundary between the plate and the body is determined from (11) and (12), assuming, respectively, that z = 0 and z = h:

at the plate surface

$$T(0, t) = \frac{2\vartheta\sqrt{t}}{\sqrt{\pi}\xi_1} \sum_{k=0}^{\infty} (-\varkappa)^k [I(m_{10}) - \varkappa I(m_{20})], \tag{17}$$

$$m_{10} = \frac{hk}{\sqrt{a_1 t}}, \qquad m_{20} = \frac{h(k+1)}{\sqrt{a_2 t}}; \tag{18}$$

at the boundary between the plate and the body

$$T(h, t) = \frac{4\vartheta\sqrt{t}}{\sqrt{\pi}(\xi_1 + \xi_2)} \sum_{k=0}^{\infty} (-\varkappa)^k I(m), \tag{19}$$

$$m = \frac{(2k+1)h}{2\sqrt{a_1 t}}. \tag{20}$$

Figure 2 shows the curves giving the rise in temperature as a function of heat-source operation at the surface of the plate in a graphite–copper system, with the thickness of the graphite plate h = 10 and 20 μm, at the boundary between the plate and the body, and within the body, at a depth of z = 100 μm from the surface of the plate.

In actual practice, the change in heat-source intensity can frequently be represented, in approximate terms, in rake-like form, as shown in Fig. 3. The height of each tooth may vary; however, within any given time interval $t_{n+1} - t_n$, $\vartheta_n = \text{const}$. The duration of the time interval may also vary. The temperature field for such a source can be determined from (11) and (12), using the Duhamel theorem. We find:

in the plate $0 \leq z \leq h$

$$T_1(z, t) = \frac{2}{\sqrt{\pi}\,\xi_1} \sum_{n=1}^{n=m} (\vartheta_n - \vartheta_{n-1}) \sqrt{t - t_{n-1}} \sum_{k=0}^{\infty} (-\varkappa)^k [I(m_{1n}) - \varkappa I(m_{2n})], \tag{21}$$

in the body $h \leq z \leq \infty$

$$T_2(z, t) = \frac{4}{\sqrt{\pi}\,(\xi_1 - \xi_2)} \sum_{n=1}^{n=m} (\vartheta_n - \vartheta_{n-1}) \sqrt{t - t_{n-1}} \sum_{k=0}^{\infty} (-\varkappa)^k I(m_{3n}), \tag{22}$$

where n is the interval number; m is the number of intervals; $\vartheta_0 = 0$ and $t_0 = 0$; t_n is the time at the end of $t - t_{n-1} = 0$, we have $I(m) = 0$;

$$m_{1n} = \frac{2hk + z}{2\sqrt{a_1(t - t_{n-1})}}, \tag{23}$$

$$m_{2n} = \frac{2h(k+1) - z}{2\sqrt{a_1(t - t_{n-1})}}, \tag{24}$$

$$m_{3n} = \frac{h(2k+1)}{2\sqrt{a_1(t-t_{n-1})}} + \frac{z-h}{2\sqrt{a_2(t-t_{n-1})}} \quad . \tag{25}$$

The temperature at the surface of the plate is

$$T(0, t) = \frac{2}{\sqrt{\pi}\,\xi_1} \sum_{n=1}^{n=m} (\vartheta_n - \vartheta_{n-1}) \sqrt{t - t_{n-1}} \sum_{k=0}^{\infty} (-\varkappa)^k [I(m_{10n}) - \varkappa I(m_{20n})] \tag{26}$$

and at the boundary between the plate and the body it is

$$T(h, t) = \frac{4}{\sqrt{\pi}\,(\xi_1 + \xi_2)} \sum_{n=1}^{n=m} (\vartheta_n - \vartheta_{n-1}) \sqrt{t - t_{n-1}} \sum_{k=0}^{\infty} (-\varkappa)^k I(m_{hn}), \tag{27}$$

where

$$m_{10n} = \frac{hk}{\sqrt{a_1(t-t_{n-1})}}, \tag{28}$$

$$m_{20n} = \frac{h(k+1)}{\sqrt{a_1(t-t_{n-1})}}, \tag{29}$$

$$m_{hn} = \frac{(2k+1)h}{2\sqrt{a_1(t-t_{n-1})}} \, . \tag{30}$$

The calculations based on (26) and (27) are simplified by introducing negative sources and using the method of superposition. Then, having calculated the curves from (17) and (19) for $\vartheta = 1$, for each of the sources we can obtain the temperature as a function of time by multiplying these curves by ϑ_n. Positioning the origin for the curve of each of the sources on the graph at the instant at which it begins to operate and summing all of the curves, with consideration of their signs, we will find the final temperature curve. This is illustrated in Fig. 3.

Figure 3a shows a rake-shaped complex energy pulse. It is replaced by 6 infinite sources with constant intensities, which begin to operate with a delay relative to each other. Thus, at the instant of time $t = 0$ the first source begins to operate with an intensity ϑ_1. At the instant $t = t_1$ the second negative source is actuated, and it exhibits the intensity $\vartheta_2 - \vartheta_1$. Then, when $t = t_2$ the third positive source is actuated, exhibiting the intensity $\vartheta_3 - \vartheta_2$, etc. At the instant $t = t_5$ the last negative source is actuated, and its intensity is $\vartheta_6 - \vartheta_5$, and here $\vartheta_6 = 0$.

The change in temperature at the plate surface as a result of the operation of the sources is shown in Fig. 3b. The change in temperature resulting from the action of the first source ϑ_1 yields the curve 1; the action of the second source $\vartheta_2 - \vartheta_1$ gives us curve 2, etc. Curve 2 is curve 1, reduced by a factor of $(\vartheta_2 - \vartheta_1)/\vartheta_1$. Curve 3 repeats curve 1, but with the ordinates increased as $(\vartheta_3 - \vartheta_2)/\vartheta_1$. The curve between t_2 and t_3 is the algebraic sum of curves 1, 2, and 3. The remaining curves have been plotted in this manner.

The use of the superposition method markedly simplifes the calculation, since with these formulas we have to calculate only a single curve, despite the complex shape of the heat source.

NOTATION

- a is the coefficient of thermal diffusivity;
- c is the heat capacity;
- h is the plate thickness;
- t is the time;
- T is the temperature;
- z is the instantaneous coordinate;
- γ is the density;
- λ is the thermal conductivity;
- ϑ is the heat-flow intensity.

Subscripts and Superscripts

1 is the plate;
2 is the body.

LITERATURE CITED

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2. A. V. Luikov, The Theory of Heat Conduction [in Russian], GITTL (1952).